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CLOSED 2-CELL EMBEDDINGS IN THE PROJECTIVE PLANE

BY

D.W. BARNETTE

Department of Mathematics
University of California
Davis, CA 95616-8633, USA

ABSTRACT

An embedding of a multi-graph in a manifold is a closed 2-cell embedding provided the closures of the faces are all closed 2-cells. In this paper we characterized the projective planar multi-graphs that have closed 2-cell embeddings in the projective plane.

1. Introduction

The **open faces** of a graph G embedded in a surface S are the connected components of $S - G$. The **closed** faces are the closures of the open faces. Three important classes of embedding of graphs are the **2-cell embeddings** in which all open faces are open 2-cells, **closed 2-cell embeddings** in which each closed face is a closed 2-cell, and the **polyhedral embeddings** in which all closed faces are closed 2-cells, each vertex is at least 3-valent and intersection of any two closed faces is connected.

The 2-cell embeddings are thus those where no face is multiply connected, the closed 2-cell embedding are those where no closed face is multiply connected and the polyhedral embeddings are those where vertices have valence at least three and no two closed faces have a multiply connected union.

We shall consider embeddings in the projective plane. The projective planar graphs (i.e. those embeddable in the projective plane) having a 2-cell embedding are easily characterized as the projective planar graphs that contain a circuit.

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In [3] the author finds a relatively simple characterization of the projective planar graphs that have a polyhedral embedding. Surprisingly, the intermediate case — the closed 2-cell embeddings — appears to be much more difficult to characterize. In this paper we give a characterizaton of the projective planar graphs and multi-graphs that have closed 2-cell embeddings in the projective plane.

2. Basic definitions and notation

The **graphs** in this paper are without loops or multiple edges. When multiple edges are to be allowed we use the term **multi-graph**. We shall use Π to denote the plane and Π^* to denote the projective plane. When referring the faces of a graph we shall use the term **face** for closed face and also for the circuit bounding it. It will be clear from the context which meaning we use. When referring to faces we will mean "open face" only when "open" is explicitly stated.

We shall use the term **CTC-embedding** for closed 2-cell embedding. Note that in a CTC-embedding the boundary of each face is a simple circuit in the graph, while in a 2-cell embedding each closed face is toplogically a polygon with identifications of vertices and edges.

If a graph G is embedded in a subset of Π^* that is a 2-cell we say that the embedding of G in Π^* is **planar**, otherwise the embedding is **nonplanar**. In particular, a circuit in a graph G embedded in Π^* is a **planar circuit** if it is homotopically trivial, otherwise it is a **nonplanar circuit**.

By a **path** in a graph we shall always mean a non-selfintersecting path. If P is a path in G then the set consisting of P minus its endpoints is the **open path** P.

A graph G is n-**connected** provided G has at least $n+1$ vertices and the graph cannot be separated by removing fewer than n vertices.

If G is a multi-graph then the graph G' formed by removing all but one edge of each set of multiple edges is called the **underlying graph** of G. If H is a subgraph of a graph G then the **complement** $\bar{H}$ of H in G is the graph consisting of all eges in G but not in $\bar{H}$ and all vertices of these edges. The **vertices of attachment** of H are the vertices in $H \cap \bar{H}$.

If G is a 2-connected graph and removing vertices a and b separates G then each connected component of $G-\{a,b\}$ is called a **2-component** of G. If C is a 2-component of $G-\{a,b\}$ then the subgraph of G consisting of C and all edges from C to a and b is a **2-piece** of G. A 2-piece of G is **minimal** if it does not properly contain any 2-piece of G. Note that if A is a 2-piece of G there will be another 2-piece in $\bar{A}$ with the same vertices of attachments as A.

For any graph G embedded in a surface, if we remove an edge $e = xy$ and add a 2-piece A such that x and y become the vertices of attachment of A, and A is planar embedded in the face of $G - e$ that contains e, we say that the new graph is obtained from G by **replacing** e **by a 2-piece**. We say G_1 is obtained from G by **replacing edges** provided $G = G_1$ or G_1 is obtained from G by repeating application of the above process.

If G is a graph embedded in a surface and x is a vertex of G then the **star** of x, denoted star(x) is the union of the closed faces meeting x. The **antistar** of x, denoted ast(x) is the union of the closed faces missing x, and the **link** of x, denoted link(x) is the intersection of the star and antistar of x.

3. Preliminary lemmas

The planar 3-connected graphs are isomorphic to the graphs formed by the vertices and edges of convex 3-dimensional polytopes with the faces of the polytope corresponding to the faces of the graph [6]. Well-known consequences of this are given in the following:

LEMMA 1: *If G is a planar 3-connected graph then*

(1) *The antistar of each vertex is a 2-cell*
(2) *The link of each vertex is a simple circuit*
(3) *If two faces meet on vertices x and y then xy is an edge of both faces.*

We say that a graph G_1 embedded in a surface S is obtained from a graph G in S by **face splitting** provided G_1 is obtained by adding an edge e to G such that e lies in a closed face of G and the vertices of e are either vertices of G or points in the relative interiors of edges of G.

We shall use the following theorem of the author [1].

LEMMA 2: *The closed 2-cell embeddings in Π^* can be generated from the embeddings $\mathcal{G}_1$ and $\mathcal{G}_2$ (see Fig. 1) by face splitting.*

Another lemma of the author [2] we shall need is

LEMMA 3: *Let G be a graph embedded in a closed cell bounded by a circuit C of G. Let C be the union of four paths $\Gamma_1, \ldots, \Gamma_4$ such that $\Gamma_i \cap \Gamma_{i+1}$ is a vertex* *and $\Gamma_4 \cap \Gamma_1$ is a vertex (we do not rule out some of the Γ_i's being single vertices). If no face of G meets both Γ_1 and Γ_3 then there is a path P joining a vertex of the open path Γ_2 to a vertex of the open path Γ_4 such that P meets C only at its endpoints.*

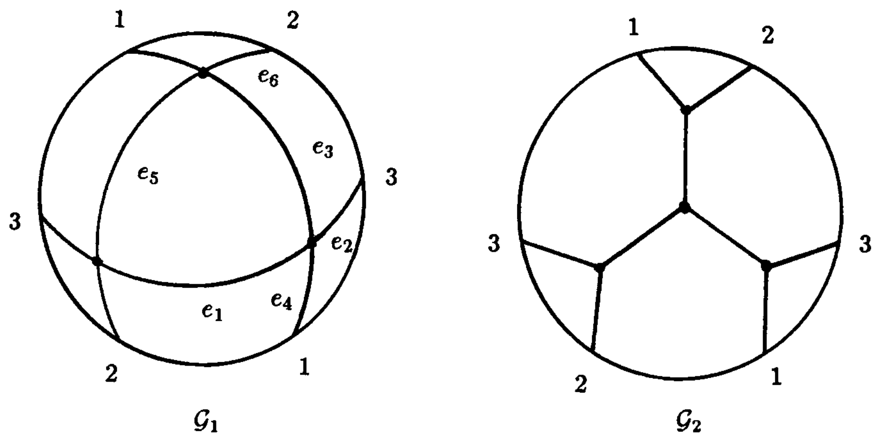


Fig. 1

Finally we shall use the following theorem of Tutte [5].

LEMMA 4: *For any planar 3-connected graph G, the faces of G are the nonseparating circuits.*

Here, a circuit is **nonseparating** provided the (topological) complement of the circuit in G is connected. So, for example, a nonseparating circuit cannot intersect an edge on just two vertices.

A **3-chain** in a planar graph G is a set of three faces F_1, F_2 and F_3 such that each two faces meet. If no vertex belongs to all three faces then the chain is **nontrivial.** When G is 3-connected and the chain is nontrivial, a 3-chain is **simple** provided the intersection of each two faces is a vertex or an edge, it is **pure** if each two faces meet on one vertex. When G is 3-connected and the 3-chain is nontrivial the complement of $F_1 \cup F_2 \cup F_3$ in Π will be two open connected sets, one bounded and one unbounded. These two sets will be called the **regions of the 3-chain.** The boundary of each region will be the union of three paths one on each of F_1,F_2 and F_3. A **triad** is a subgraph of G consisting of a vertex x in one of the regions of the 3-chain together with three paths, each joining x to one of the three open paths on the boundary of the region containing x. If there is a triad in each of the two regions of the 3-chain we say that the 3-chain has a **triad pair**.

LEMMA 5: *If G has a closed 2-cell embedding in Π^* then G is 2-connected.*

Proof: Clearly G is connected. Since each face is bounded by a simple circuit, the link of any vertex is connected. Suppose removing a vertex x disconnects G. Since link(x) is connected, one component, say C_1 of $G - x$ misses link(x). Now however, since x is joined only to link(x), C_1 is a component of G and G is not connected, a contradiction. ∎

LEMMA 6: *Let G be a 2-connected graph embedded in Π^*. Let H be a subgraph of G whose induced embedding in Π^* is a CTC-embedding. Then G is a CTC-embedding.*

Proof: If any open face F of G is multiply connected, then since F lies in a face of H, F will separate connected components of G, a contradiction. Thus the embedding is a 2-cell embedding. It now follows that topologically, the closed faces are polygons with possible identifications of vertices and edges. If there are any identifications of vertices of a face F then in F there is a simple closed curve meeting the boundary of F at an identified vertex x of F and separating the boundary of F. Thus removing x separates G, a contradicion. Thus all faces are closed 2-cells and G is CTC-embedded in Π^*. ∎

4. Embedding planar graphs in Π^*

THEOREM 1: *If G is a planar 2-connected graph with a nontrivial simple 3-chain without a triad pair then G has a CTC-embedding in Π^*.*

Proof: First we show that G embeds in Π^*, then we prove that it is a CTC-embedding. Let F_1, F_2 and F_3 be the faces of the 3-chain. We shall treat the case where each two faces of the chain intersect on a single vertex. The other cases are similar.

Figure 2 shows the 3-chain and Figure 3 shows how we will embed the vertices and edges of the 3-chain in Π^*. We shall assume that the region A in Figure 2 is a region without a triad. The region of the planar embedding bounded by $P_4 \cup P_5 \cup P_6$ can be embedded in the cell in Π^* bounded by $P_4 \cup P_5 \cup P_6$. We embed the subgraph S of $G - (F_1 \cup F_2 \cup F_3)$ that lies in A as follows.

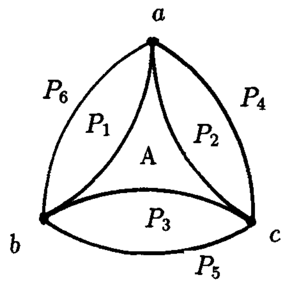


Fig. 2

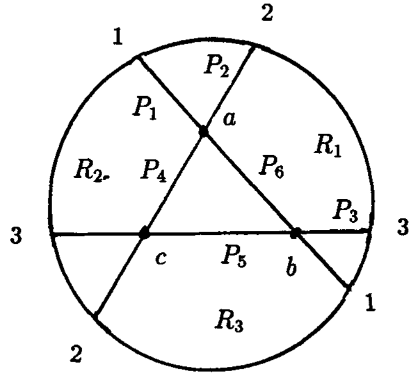


Fig. 3

Let x be a vertex of S. We shall say that x **connects to** P_i provided there is a path in S from x to a vertex of the open path P_i.

Let G_1 be a maximal subgraph of S such that no vertex of G_1 connects to P_1. Let G_2 be a maximal subgraph of $S - G_1$ such that no vertex of G_2 connects to P_2 and let G_3 be a maximal subgraph of $S - (G_1 \cup G_2)$ such that no vertex of G_3 connects to P_3. Clearly we can embed each G_i in region R_i (see Fig. 2). We need to show that $G_1 \cup G_2 \cup G_3 = S$.

Suppose x is a vertex of S not in $G_1 \cup G_2 \cup G_3$ then x connects to the open paths P_1,P_2 and P_3 and there is a triad in region A, a contradiction.

Suppose e is an edge of S not in $G_1 \cup G_2 \cup G_3$. The edge e cannot join vertices of two different G_i's (for example, if it joined a vertex of G_1 to a vertex of G_3 then every vertex of G_1 connects to P_1, a contradiction). By maximality of G_i,

e doesn't join two vertices of G_i. By maximality, e doesn't join any G_i to a path P_j for $j \neq i$. By the definition of the G_i's, e doesn't join any G_i to the open path P_i. By maximality, e does not join a G_i to a,b or c. Finally, if e joins two P_i's then it is in one of the G_i's by their definition. Thus $G_1 \cup G_2 \cup G_3 = S$ and we have embedded G in Π^*.

We now observe that our embedding of $F_1 \cup F_2 \cup F_3$ in Π^* is a closed 2-cell embedding and thus by Lemma 6 we have a CTC-embedding of G. ∎

LEMMA 7: *Let G be a planar 3-connected graph. If F_1,F_2 and F_3 form a nontrivial 3-chain with a triad pair then at least one of F_1, F_2, or F_3 is a planar circuit in any embedding of G in Π^*.*

Proof: We first treat the case where the 3-chain is pure. Figure 4 shows the embedding of $F_1 \cup F_2 \cup F_3$ where each is a nonplanar circuit. One of the triads will be in R_1 but the other must be in R_2,R_3 or R_4. This however is impossible, for example if the triad is in R_2 it can be connected to the open paths e_1 and e_2 but not the open path e_3.

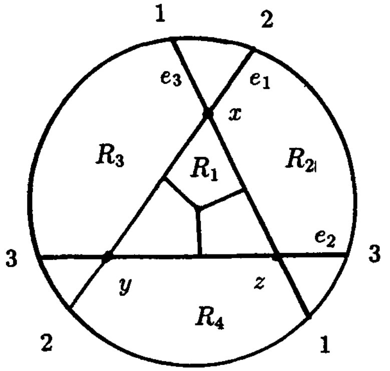


Fig. 4

If any pairs of the F_i's meet on edges we take the embedding in Π^* and shrink the edges of intersection to vertices, giving us case I. (Note that since G is 3-connected two faces of the chain can meet only on one vertex or edge.) ∎

LEMMA 8: *If G is a planar 3-connected graph in which there are four faces F_1,F_2,F_3 and F_4 such that each three meet at a vertex, then G is K_4 the complete graph on four vertices.*

Proof: Let v_i be the vertex in $\bigcap_{j \neq i} F_i$. Then by Lemma 1 each two vertices v_i and v_k are joined by the edge $e_{i,k} = \bigcap_{j \neq i,k} F_j$. Thus G contains K_4. But now each F_i contains three of the edges $e_{j,k}$ and these three edges form a face of the embedding of K_4 thus F_i is a face of the subgraph isomorphic to K_4 (for each i). It follows that $G = K_4$. ∎

We shall say that a planar graph is **Π*-CTC-embeddable** provided it has CTC-embedding in Π^*.

THEOREM 2: *The planar 3-connected* Π^**-CTC-embeddable graphs are* K_4 *and the planar 3-connected graphs with nontrivial 3-chains without triad pairs.*

Proof: The graph $\mathcal{G}_2$ in Fig. 1 is a CTC embedding of K_4 in Π^*. Theorem 1 gives the embeddings of the others.

To see the necessity of the conditions, let G be a planar 3-connected Π^*-CTC-embeddable graph. Let S be the set of faces of the embedding of G in Π that are not faces in Π^*. Since any planar circuit in Π^* that is not a face will separate the graph and since faces in Π are nonseparating, we see that all faces in S are nonplanar circuits in Π^*. It follows that each two faces in S have a vertex in common.

If S contains at least four faces then each three meet at a vertex, thus $G = K_4$ and we are done. Thus either G contains three faces not meeting at one vertex or all faces in S meet at a vertex x. In the first case the three faces form a nontrivial 3-chain.

In the second case the antistar of x consists of faces in Π that are faces in Π^*, and the antistar of x in Π is embedded as a cell in Π^*. In the complement of $\mathrm{ast}(x)$ in Π^* we have the vertex x and edges from x to $\mathrm{link}(x)$. It is easily seen that no matter how x is joined to $\mathrm{link}(x)$ there will be a face meeting x that is not a closed 2-cell.

Since the nontrivial 3-chain we have obtained in this case consists of faces that are not planar circuits in Π^*, Lemma 7 implies that the 3-chain does not have a triad pair. ∎

THEOREM 3: *A planar 3-connected multigraph G is* Π^**-CTC-embeddable if and only if the underlying graph G' is* Π^**-CTC-embeddable.*

Proof: The sufficiency of Π^*-CTC-embeddability of the underlying graph is obvious. Suppose G is Π^*-CTC-embeddable. If e_1, e_2 is a pair of edges with endpoints x and y in the embedding in Π we shall show that removing either e_1 or e_2 creates another CTC-embedding.

CASE I: $e_1 \cup e_2$ is a planar circuit. In this case since one face, F_1, meeting e_1 lies inside $e_1 \cup e_2$ and the other, F_2, lies outside $e_1 \cup e_2$, $F_1 \cup F_2$ cannot have a multiply connected union and we may remove e_1.

CASE II: No pair of multiple edges forms a planar circuit but $e_1 \cup e_2$ is a nonplanar circuit. Suppose we cannot remove e_1 and thus the faces, F_1 and F_2, containing e_1 have a multiply connected union. Because of edge e_2 making a nonplanar circuit with e_1, $F_1 \cup F_2$ cannot lie in a subset of Π^* that is a cell. It follows that $F_1 \cup F_2$ contains a simple close curve Γ_1 that is not contractible in Π^*. Similarly if we cannot remove e_2 then the union of the two faces, F_3 and F_4, containing e_2 contains a simple closed curve Γ_2 that is not contractible in Π^*. Now Γ_1 and Γ_2 must meet at a point p which lies in F_1,F_2,F_3 and F_4. Since G is 3-connected and has no multiple edges forming planar circuits, we have that F_1 and F_3 meet on an edge, say xp. Now F_2 and F_3 meet on yp, F_1 and F_4 meet on yp and F_2 and F_4 meet on xp. Now the entire graph consists of the three vertices x,y,p and the double edges xp,yp, and xy. This contradicts the 3-connectedness of G since G does not have at least four vertices.

It follows that we may remove either e_1 or e_2. By continuing to remove such edges we eventually arrive at a Π^*-CTC-embedding of G'. ∎

THEOREM 4: *If G is a 2-connected planar Π^*-CTC-embeddable multi-graph then G has a pure 3-chain, without a triad pair in an embedding in Π or G is obtained from a Π^*-CTC-embeddable 3-connected multi-graph by replacement of edges.*

Proof: By a theorem of the author [1] the CTC-embeddings in Π^* can be generated by face splitting from the two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ embedded as shown in Fig. 1.

CASE I: We can generate G from $\mathcal{G}_1$.

In the planar embedding of $\mathcal{G}_1$, e_1 and e_2 bound a face F_1, e_3 and e_4 bound a face F_2 and e_5 and e_6 bound a face F_3. When we split faces to construct G, each of the edges e_i becomes a path E_i. From the embedding of $\mathcal{G}_1$ in Π^* we see that it is impossible to have a path from the open path E_1 to the open path E_2 missing the other E_i's. Thus by Lemma 3, a face F_1' lying in F_1 meets a and b. Similarly a face F_2' in F_2 meets b and c and a face F_3' in F_3, meets c and a. These three faces form a pure nontrivial 3-chain in G.

Any triad pair for F_1',F_2', and F_3' would have to be a triad pair for F_1, F_2 and F_3 but such a triad pair cannot exist for the embedding of $\mathcal{G}_1$ in Π^*.

CASE II: We generate G from $\mathcal{G}_2$. We prove this case by induction on the number of edges of G starting the induction with $G = \mathcal{G}_2$.

Now suppose $G \neq \mathcal{G}_2$. If G is 3-connected then there is nothing to prove thus we assume G is 2- but not 3-connected.

Splitting faces of $\mathcal{G}_2$ results in possible vertices being added to the edges of $\mathcal{G}_2$ thus in G there is subgraph H consisting of the edges of $\mathcal{G}_2$ with possible vertices added. We separate G by removing two vertices x and y. If some 2-component of G contains H then some other 2-component can be chosen not containing H. Let A be a 2-piece obtained from a 2-component not containing H.

One may easily check that if we choose two vertices of H that do not lie on one of the original edges of $\mathcal{G}_2$ then we cannot separate these two vertices from any other vertex of H by removing fewer than three vertices of H. Thus if $A - \{x, y\}$ contains vertices of H they all lie on one edge of $\mathcal{G}_2$. Suppose $A - \{x, y\}$ contains vertices of H lying on an edge e of $\mathcal{G}_2$ (see Fig. 5). It is also easily seen that vertices on e cannot be separated from other vertices of H by removing fewer than three vertices unless two vertices on e are removed. It follows that the vertices of attachments of A are on e.

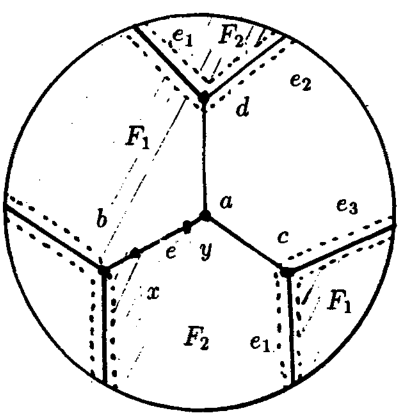


Fig. 5

We may now choose neighborhoods N_1,N_2 and N_3 of edges e_1, e_2 and e_3 that miss A as shown in Fig. 5. Now A lies in the region $F_1 \cup F_2 - (N_1 \cup N_2 \cup N_3)$. Thus A lies in a closed 2-cell E in Π^*. Since A is a connected graph lying in E

and meeting x and y we can contract A to an edge joining x and y. Since the contraction can be done in the closed cell E, only faces lying in E could have their topological type changed.

We claim that the graph G_1 produced by the contraction is CTC-embedded. If a face F is multiply connected after the contraction then it must meet A on at least two vertices and if it does not meet A on a vertex other than x or y then it doesn't become multiply connected. Let z be a vertex of $F \cap A$ that is not x or y.

If $\{x, y\} \cap F = \emptyset$ then F is contracted to a point. If $\{x, y\} \cap F = x$ or y then F will be contracted to x or y. If $\{x, y\} \cap F = \{x, y\}$ then the path Γ from x to y along F containing z contracts to the edge xy, thus F remains a cell if $F - \Gamma$ misses A. If $F - \Gamma$ meets A then F contracts to xy. Thus F does not become multiply connected.

Now by induction, G_1 has a pure nontrivial 3-chain F_3,F_4,F_5 or G_2 is obtained from a Π^*-CTC-embeddable 3-connected multi-graph by replacing edges. In the second case G is also of the desired type.

In the first case, replacing the edge xy by a 2-piece meeting x and y in Π does not change the vertices of intersection of the faces F_3,F_4 and F_5 in Π, thus G has a nontrivial pure 3-chain. Clearly replacing the edge xy by a 2-piece does not create a triad pair for F_3,F_4 and F_5.

If $A - \{x, y\}$ does not meet H then A lies in one face of H in Π^* and the above argument about contradicting A to an edge holds. ∎

The previous Theorems give us the following characterization for planar graphs.

THEOREM 5: *A planar multi-graph is Π^*-CTC-embeddable if and only if it is 2-connected and*

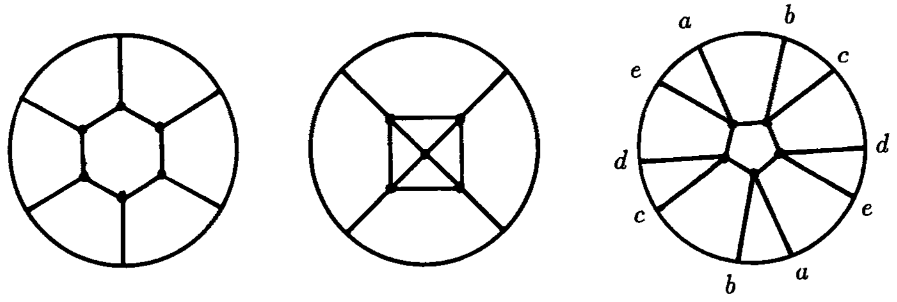
(1) *has a pure 3-chain without a triad pair, or*
(2) *is obtained from a multi graph, whose underlying graph is K_4, by replacing edges, or*
(3) *is obtained from a 3-connected multi-graph with a nontrivial 3-chain without a triad pair, by replacing edges.*

5. Embedding nonplanar graphs in Π^*

We now turn to the nonplanar Π^*-CTC-embeddable graphs.

By a famous theorem of Kuratowski [4] every nonplanar graph contains a refinement of K_5 or the complete bipartite graph $K_{3,3}$. Fig. 6 shows the two embeddings of K_5 and the only embedding of $K_{3,3}$ in Π^*. Since there are CTC-embeddings, by Lemma 6 we have

THEOREM 6: *Every embedding of a 2-connected nonplanar multi-graph in* Π^* *is a CTC-embedding.*



$K_{3,3}$ K_4

Fig. 6

This completes the characterization of multi-graphs with closed 2-cell embeddings in Π^*.

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